13/9/22

Tutorial Planning

Assignment Notes:

1) Free to choose either sign convention $B' = \tau N$ or $B' = -\tau N$ as long as you tell me which you are using for which problem and you stay consistent through your solution for that problem. In particular for Q3, β defined assuming $B' = \tau N$ sign convention.

2) For Q5, you are free to assume circular helix is given by a parametrization of a certain form

$$\alpha(t) = (a\cos t, a\sin t, b).$$

You can then either argue using Fund. Thm. of curves or solve the ODE problem of Frenet formulas to recover the above param. You are also free to adapt the definition of cylindrical helix to give a "general" definition for circular helix

$\langle T, u\rangle = \cos\theta_0 = \text{const.}$ and trace of α projected onto TN plane is a circle.

Recall Def: Differential of a map: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. Then the differential of f at $p \in U$,

$$df_p : \mathbb{R}^n \to \mathbb{R}^m$$

is defined as follows: Let $\alpha : (-\varepsilon, \varepsilon) \to U$ be a smooth curve with $\alpha(0) = p$, $\alpha'(0) = w$, then the curve $\beta = F \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ is smooth and we define df_p by

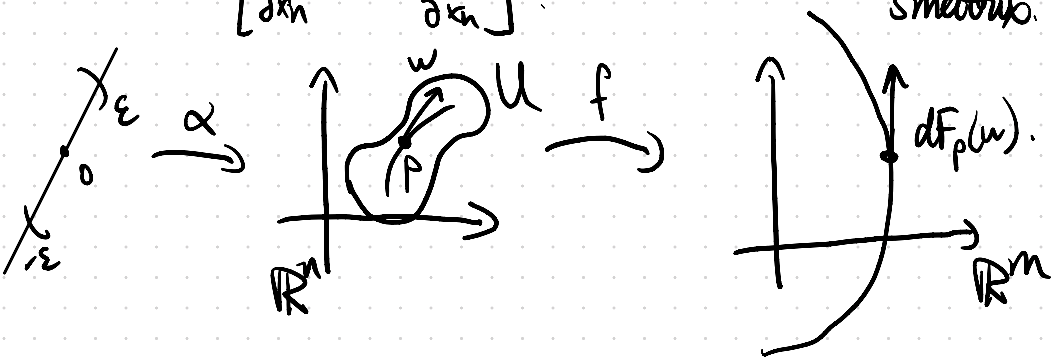
$$df_p(w) = \beta'(0).$$

Note, when written in the standard basis, $\{e_1, \dots, e_n\}$, df_p is the Jacobian matrix of f at p. i.e.

$$df_p = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n.}}$$

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) = \frac{\partial (f_1, \dots, f_n)}{\partial (x_1, \dots, x_n)}$$

for a square matrix.

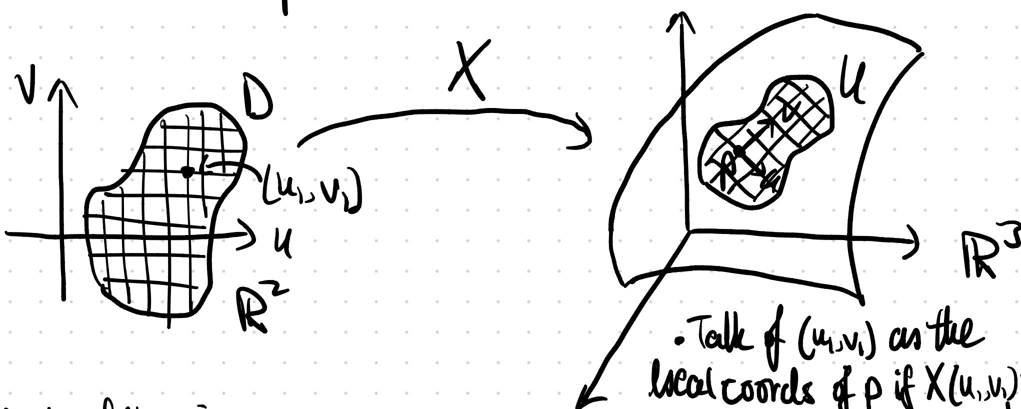


- $df_p : \mathbb{R}^n \to \mathbb{R}^m$

these are tangent spaces at p, $f(p)$.

<u>Recall Def</u>: (Regular Surface): $M \subset \mathbb{R}^3$ is a regular surface if for each $p \in M$, there is a nbhd $U \subset \mathbb{R}^3$ and an open set $D \subset \mathbb{R}^2$ and a map $X : D \to U \cap M$ s.t.

1) X is smooth
2) dX is full rank: $X_u = \frac{\partial X}{\partial u}$, $X_v = \frac{\partial X}{\partial v}$ are linearly independent for any $(u,v) \in D$ ($\Leftrightarrow dX_p$ is 1-1 for each $p \in D$).
3) X is a homeomorphism

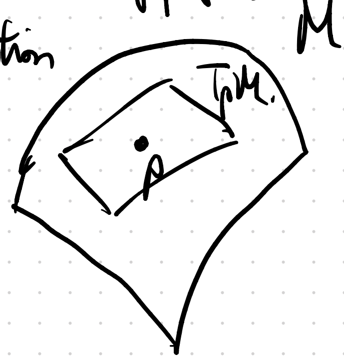


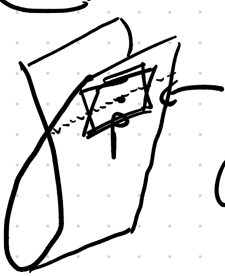
• Talk of (u_1,v_1) as the local coords of p if $X(u_1,v_1)=p$.

Avoid the following:

← no tangent plane here. (regularity condition)

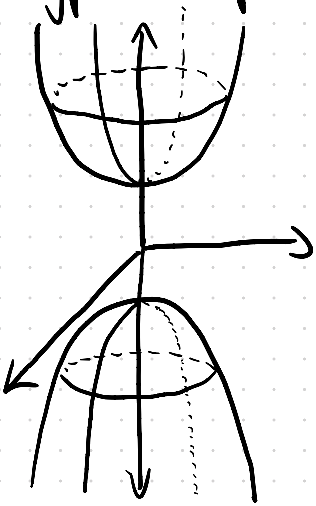
Regularity condition (condition 2).





← also no tangent plane here. (homeomorphism condition).

Ex 1: Hyperboloid of 2 Sheets: $-x^2-y^2+z^2=1$. Show that this is a regular surface and find a parametrization.



Define $f(x,y,z) = -x^2-y^2+z^2-1$.

Clearly the surface is the inverse image $f^{-1}(0) = \{(x,y,z) : -x^2-y^2+z^2=1\}$.

Clearly $f: \mathbb{R}^3 \to \mathbb{R}$ is smooth, and 0 is a regular value of f since

$\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial y} = -2y$, $\frac{\partial f}{\partial z} = 2z$. So ∇f vanishes only at $(0,0,0)$ and $(0,0,0) \notin f^{-1}(0)$. So it is a regular surface.

Rewrite as $x^2+y^2+1=z^2$. Taking $r^2=x^2+y^2$ (i.e. $x=r\cos v$, $y=r\sin v$)

Then we get $r^2+1=z^2 \iff 1=z^2-r^2$.

Then by hyperbolic trig identity $\cosh^2 u - \sinh^2 u = 1$ we take $z=\cosh u$, $r=\sinh u$, then we have

$(x,y,z) = (\sinh u \cos v, \sinh u \sin v, \cosh u)$.

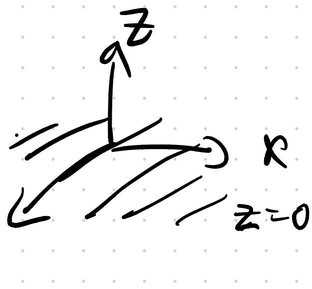
This is an example of a regular surface that is disconnected

Ex2: $f(x,y,z) = z^2$. Show $f^{-1}(0)$ is a regular surface.

$f^{-1}(0) = \{(x,y,z) : z^2 = 0\} \iff \{z : z = 0\}$.

0 is not a regular value of f since

$\nabla f = (0, 0, 2z)$, so ∇f vanishes when $z = 0$, ie. $0 \in f^{-1}(0)$.



By above, $f^{-1}(0)$ is the plane $z = 0$

We'll show directly from the definition that this is a regular surface.

Let $p \in f^{-1}(0)$, then write $p = (u, v, 0)$. for some $u, v \in \mathbb{R}$.

we have specified a param. of $f^{-1}(0)$.

So $X : \mathbb{R}^2 \to f^{-1}(0)$ by $X(u,v) = (u, v, 0)$.

- X is clearly smooth, • a homeomorphism of $\mathbb{R}^2$ with $f^{-1}(0) \cong \mathbb{R}^2$.

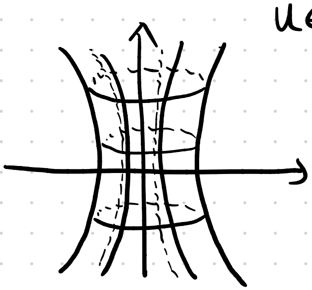
$$dX = \left(\frac{\partial X_i}{\partial u_j}\right) = \begin{bmatrix} \frac{\partial X_1}{\partial u} & \frac{\partial X_1}{\partial v} \\ \frac{\partial X_2}{\partial u} & \frac{\partial X_2}{\partial v} \\ \frac{\partial X_3}{\partial u} & \frac{\partial X_3}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

columns are lin. indep.

So • $f^{-1}(0)$ is a regular surface.

Ex 3: Catenoid: $(c\cosh\frac{v}{c}\cos u,\ c\cosh\frac{v}{c}\sin u,\ v)$

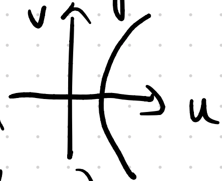
$u \in [-\pi, \pi)$, $v \in \mathbb{R}$, $c \neq 0$ const.



Is the surface of revolution obtained by rotating the catenary

$u = c\cosh\frac{v}{c}$

about the vertical.



Write the catenary as $\alpha(v) = (0, c\cosh\frac{v}{c}, v)$.

Then rotating about the z-axis, we have

$X(u,v) = (c\cosh\frac{v}{c}\cos u,\ c\cosh\frac{v}{c}\sin u,\ v)$.

Clearly X is smooth, we have restricted domains of u, v so that X is homeomorphic. Finally, we have

$$X_v = (\sinh v\cos u,\ \sinh v\sin u,\ 1)$$

$$X_u = (-c\cosh\tfrac{v}{c}\sin u,\ c\cosh\tfrac{v}{c}\cos u,\ 0).$$

which we can see are linearly independent.

The catenoid is an example of a minimal surface

(it locally minimizes surface area

$\Leftrightarrow$ critical point of the area functional $\int dA$

$\Leftrightarrow$ mean curvature identically 0 $\vec{H} := \frac{1}{2\pi}\int_0^{2\pi} k(\theta)\,d\theta \equiv 0$).